*Jour. Ind. Soc. Ag. Statistics*
*43 (1), 1991 : 9 – 15*

# MONOTONIC APPROXIMATION : THE CONCEPT AND APPLICATION TO SOIL SCIENCE

H.S. Acharya
*Marathwada Agricultural University, Parbhani*
(Received : May, 1985)


## Summary

Given $\underline{y} \in E$, a vector $\underline{z} \in E$ is defined to be a monotonic nonincreasing approximation to $\underline{y}$ if $z_{i+1} \le z_i$ , $i = 1, 2, \ldots, n-1$ and the Euclidean metric $d(\underline{z}, \underline{y})$ is minimum. Properties of such an approximation and a simple method of obtaining the same are discussed. The concept is shown to find an important application in computation of soil water diffusivity from the experimental data on moisture content in a horizontal soil column.


*Key Words* : monotonic approximation, Euclidean metric, quadratic programming, soil water diffusivity.

## *Introduction*

Consider an experiment where $y_i$, $i = 1, 2, \ldots, n$ are values of the dependent variable $y_i$ measured at values $x_i$, $i = 1, 2, \ldots, n$ of the independent variable x, which we expect to satisfy the conditions $y_{i+1} \le y_i$ , $i = 1, 2, \ldots, n-1$ but the actual measurements do not confine to this expectation. The violations of the conditions occur because of the errors arising out of uncontrollable experimental factors. One comes across such a situation while conducting an experiment to determine the soil water diffusivity by the method of Bruce and Klute [2]. In this method a cylindrical column of experimental soil, maintained in a horizontal position, is exposed to ponded water at one end (Fig. 1). After a lapse of time T hrs. the column is sectioned at constant space intervals to determine moisture contents $y_i$, $i = 1, 2, \ldots, n$ at distances $x_i$, $i = 1, 2, \ldots, n$ from the end exposed to water. Though one would expect, under homogeneous conditions, the water content to fall with increasing distance the actual values show scattering as shown in Fig. 2. This leads to mathematical difficulties in evaluating the soil water diffusivity D(y) given by

$$D(y) = -\frac{1}{2T}\frac{dx}{dy}\int_{y_0}^{y} x\,dy \qquad (1)$$

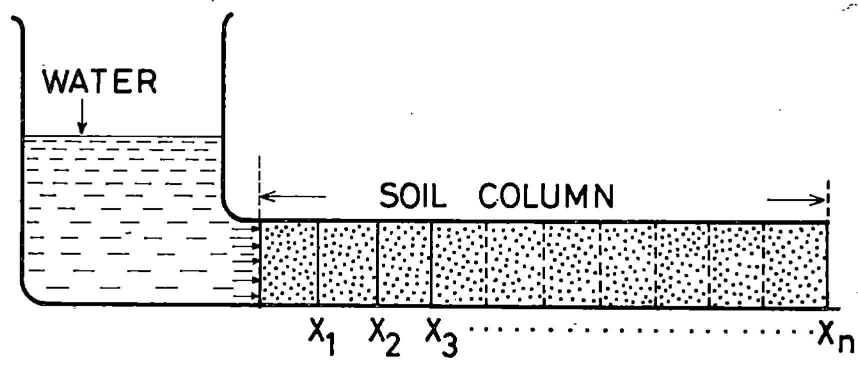


FIG. 1 : THE EXPERIMENTAL SET-UP

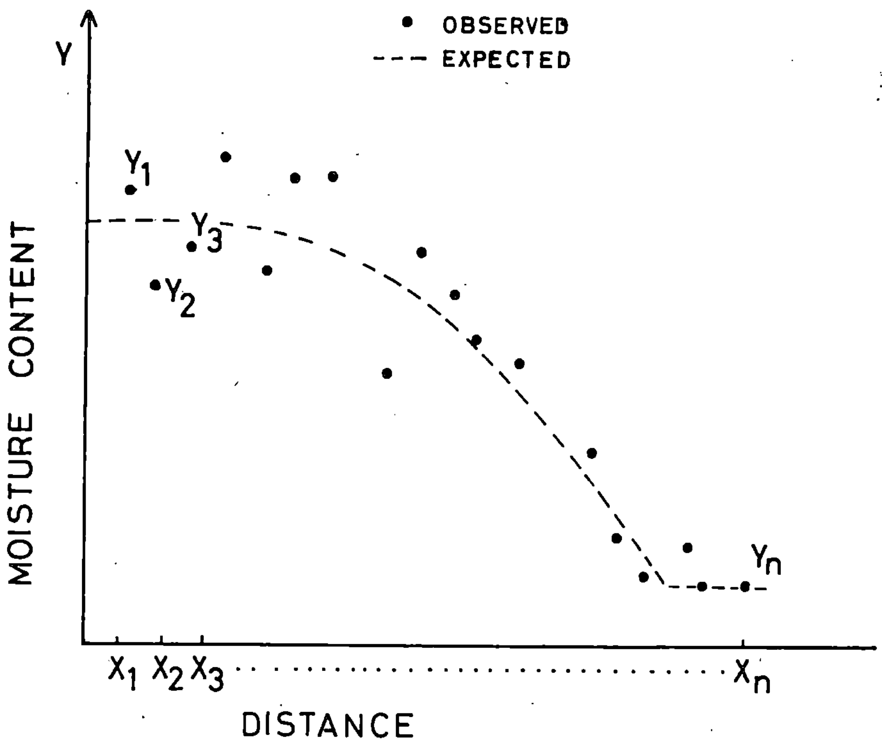


FIG. 2 : EXPERIMENTAL SCATTER OF MOISTURE

Where T is the time period, x is the space co-ordinate and $y_o$ is the initial moisure content in the soil column (Bruce and Klute, [2], Whisler *et al* [6]. The mathematical difficulties are aggravated by the fact that the x-y distribution does not yield to any of the known methods of regression to give a satisfactory curve y = y(x). This leads to uncertain ways of obtaining the values of dx/dy and $\int$ xdy in the expression for evaluation of D(y) (Selim *et al*, [4] ).

An attempt at suggesting ways to solve these problems is aimed at in the ensuing discussions. The concept of monotonic approximation which would form an important tool in this regard will be discussed to startwith. Utility of this concept in evaluating soil water diffusivity will be discussed subsequently.

The Monotonic Approximation:

*Definition 1*: Given a vector $\underline{y} = (y_1 \ \ y_2 \ \ldots y_n) \in E$

where E is the Euclidean n-space, the vector $\underline{z} \in E$ is a monotonic non increasing approximation to $\underline{y}$ if

1) $d(\underline{y}, \underline{z})$ is minimum and
2) $z_{i+1} \leq z_i$ ; i=1, 2, ..., n-1

Here 'd' is the usual Euclidean metric in E. Clearly the approximation $\underline{z}$ is to be obtained as the solution of the quadratic programming problem.

$$\text{Minimize } P = \sum_{i=1}^{n} (y_i - z_i)^2 \tag{2}$$

$$\text{Subject to } z_{i+1} - z_i \leq 0 \ ; \ i=1, 2, \ldots\ldots, n\text{-}1 \tag{3}$$

Though the Quadratic Simplex Algorithm (Wagner, [5] can be employed to obtain the solution, a probe into the properties of the monotonic approximations helps in obtaining a very simple alternative to the method.

*Theorem 1*: Given $\underline{y} \in E$, the quadratic programming problem defined by equations (2) and (3) has a unique solution.

Proof : Using the transformations

$$z_i = y_i + t_i, \ i = 1, 2, \ldots, n \tag{4}$$

the minimization problem can be reduced to

$$\text{minimize} \sum t_i^2 \tag{5}$$

subject to $t_{i+1} - t_i \le y_i - y_{i+1}$ (6)

Inequalities (6) define a finite intersection of half-spaces in E. Hence the set $F = \{ \underline{t} : \underline{t} \in E$ and $\underline{t}$ satisfies conditions (6) $\}$ is closed convex set. F is non empty, because if we let $t_i = Y - y_i$ with $Y = \max(y_1, y_2, \ldots, y_n)$ then $\underline{t} \in F$. Now since F is closed there exists a $\underline{t} \in F$ such that $d(\underline{O}, F) = d(\underline{O}, \underline{t})$ implying the existance of a solution to the minimization problem under consideration. To establish the uniqueness of the solution, let $\underline{r}$ and $\underline{s}$ be two distinct solutions. If we let $\underline{t} = (\underline{r} + \underline{s})/2$ then it can be readily shown that $\underline{t} \in F$ and $d(0, \underline{t}) < d(0, \underline{r}) = d(\underline{0}, \underline{s})$ implying a contradiction. Hence $\underline{r} = \underline{s}$.

*Theorem 2* : If $\underline{z}$ is the monotonic nonincreasing approximation to $\underline{y}$ and $y_{k+1} \ge y_k$ then $z_{k+1} = z_k$

Proof : Let $z_k > z_{k+1}$

and let $\underline{w} = (z_1\ z_2 \ldots z_{k-1}\ \bar{z}\ \bar{z}\ z_{k+2} \ldots z_n)$

where $\bar{z} = (z_k + z_{k+1})/2$

It can be readily verified that $d(\underline{w}, \underline{y}) < d(\underline{z}, \underline{y})$.

by direct computation. Hence $\underline{w}$ is an improvement over $\underline{z}$, which is a contradiction. Hence the Theorem.

*Theorem 3*: If $\underline{z}$ is the monotonic nonincreasing approximation to $\underline{y}$ and

$$y_{k+m} \ge y_{k+m-1} \ldots\ldots \ge y_k \quad \text{then}$$

$$z_{k+m} = z_{k+m-1} = \ldots = z_k$$

Proof: Follows from theorem 2.

*Definition 2* : If $\underline{z} \in E$ is the monotonic nonincreasing approximation to both $\underline{y}$ and $\underline{w}$ then we define : $\underline{y} \sim \underline{w}$.

*Theorem 4* : The relation " $\sim$ " is an equivalance relation in E.

Proof : It can be readily verified that " $\sim$ " is reflexive, symmetric and transitive. Hence the theorem.

*Theorem 5* : If $\underline{y} \in E$ is such that $y_{k+m} \ge y_{k+m-1} \ge \ldots \ge y_k$ and $\underline{y}' \in E$ is such that

$y_i = y_i{}'$ for $i < k$ and $i > k+m$ and

$$\left( \sum_{j=k}^{k+m} y_j \right) \times \frac{1}{m} = y'_i \quad \text{for } k \le i \le k+m$$

then $\underline{y} \sim \underline{y}'$. Further $d(\underline{z}, \underline{y}') \leq d(\underline{z}, \underline{y})$ if $z$ is the common monotonic nonincreasing approximation to both.

Proof : Let $z = \{\underline{r} : r_{i+1} \leq r_i\ ,\ i = 1, 2, \ldots, n ; \underline{r} \in E\}$

and let $\left(\sum_{j=k}^{k+m} y_j\right) \times \frac{1}{m} = q$

Now it can be shown that

$$d^2(\underline{r}, \underline{y}) = d^2(\underline{r}, \underline{y}') + \sum_{j=k}^{k+m} (y_j - q)^2 ; \underline{r} \in z$$

Hence we have

$$d(\underline{r}, \underline{y}) \text{ is min.} \Longleftrightarrow d(\underline{r}, \underline{y}') \text{ is min.} \quad r \in z$$

Hence $\underline{y} \sim \underline{y}'$ and $0 < d(\underline{r}, \underline{y}') \leq d(\underline{r}, \underline{y})$

The above theorem suggests a simple algorithm for calculating $\underline{z}$ given $\underline{y}$.

The algorithm:

Step 1 : Let $\underline{y} = \underline{y}^{(0)}$ be the initial approximation.

Step 2 : Using the $k^{th}$ approximation $\underline{y}^{(k)}$ the $k+1^{th}$ approximation $\underline{y}^{(k+1)}$ is to be obtained as follows:

(a) Identify the largest subset

$$y_j^{(k)}, y_{j+1}^{(k)}, \ldots, y_{j+m}^{(k)} \text{ of } \underline{y}^{(k)} = (y_1^{(k)} y_2^{(k)} \ldots y_n^{(k)})$$

which satisfies the conditions

i) $y_j^{(k)} \leq y_{j+1}^{(k)} \ldots\ldots \leq y_{j+m}^{(k)}$ and

ii) $y_j^{(k)} < y_{j+m}^{(k)}$

(b) If no subset as required in (a) exists then $\underline{y}^{(k)}$ is the required solution. Hence stop. Otherwise replace the values

$$y_j^{(k)}\ ,\ y_{j+1}^{(k)} \ldots\ldots ,\ y_{j+m}^{(k)}$$

by $\quad y_j^{(k+1)} = y_{j+1}^{(k+1)} = \ldots\ldots = y_{j+m}^{(k+1)} = \left(\sum_{i=j}^{j+m} y_i\right) \times \frac{1}{m}$

and let $\quad y_i^{(k+1)} = y_i^{(k)} \quad$ for $i < j$ and $i > j+m$

Go back to step 2(a).

It follows from Theorem 5 that $\underline{y}^{(k+1)}$ obtained as above is an improvement over $y^{(k)}$. Hence one has to start with the given $\underline{y}$ as the initial approximation and continue with Step 2 till the condition mentioned in Step 2(b) is reached.

Application to Soil Science :

Acharya, [1] has demonstrated the utility of the concept of monotonic approximation in evaluating the soil water diffusivity. As has been already mentioned in the introduction, the experimental values of the moisture contents at distances $x_i$ show considerable scatter. The main reason for this being the fact that

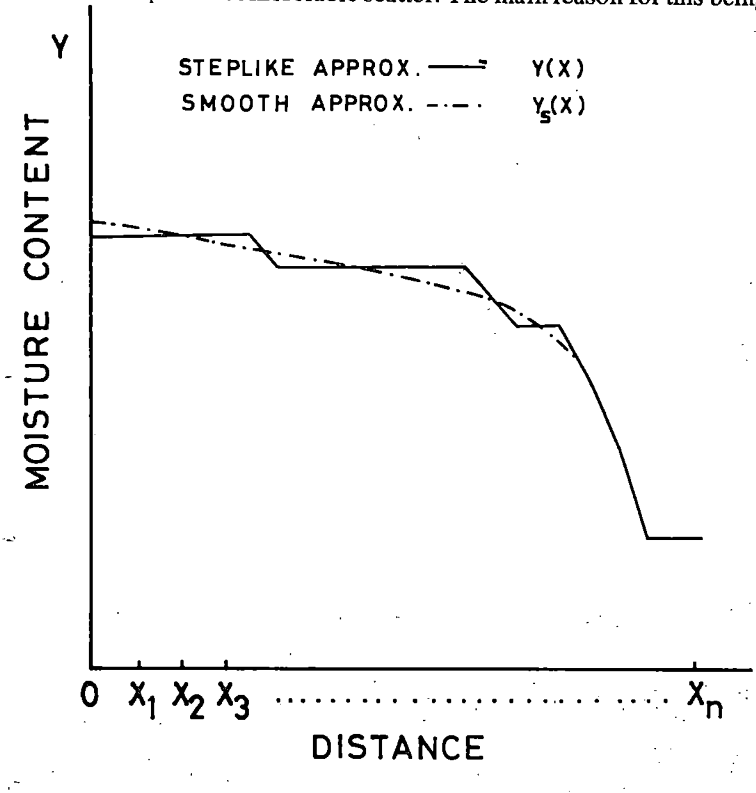


FIG. 3 : APPROXIMATIONS TO MOISTURE PROFILE

it is not possible to fill the experimental soil column with uniform bulk density. Construction of $\{y_i'\}$ as the monotonic appoximation to $\{y_i\}$ before proceeding with further calculations would eliminate the effects of nonhomogeneity to a large extent. This gives a steplike function y(x) (Fig. 3). A reasonably smooth approximation $y_s(x)$ to this can be obtained by joining the mid points of the horizontal line segments. Now the function $y_s(x)$ is enough well behaved to allow the use of numerical quadrature and numerical differentiation formula (Froberg, [3] to evaluate the differentials and integrals for use in Eqn. (1).

## REFERENCES

[1] Acharya H.S., 1982. A note on the methods for determining soil water diffusivity. Australian J. of Soil Research, 20. 61-4.

[2] Bruce, R.R. and Klute, A., 1956. The measurement of soil water diffusivity. Soil Sci. Soc. Am Proc. 20, 458-62.

[3] Froberg C.E., 1965. Introduction to numerical analysis. Addison-Wesley Pub. Con. Inc., London.

[4] Selim H.M., Kirkham, D., and Amemiya, M., 1970. A comparison of two methods for determining soil water diffusivity, Soil Sci. Soc. Am. Proc., 34, 14-18.

[5] Wagner H.M., 1974. Principles of operations research. Prentice Hall of India, New Delhi.

[6] Whisler F.D., Klute, A., and Peters, D.B., 1968. Soil water diffusivity from horizontal infiltration, Soil Sci. Soc. Am. Proc. 32, 6-11.